A Topological Construction to Solve Poincare´s Conjecture

J. Rodrigo. A. M. Díaz; rod.mart.dz@gmail.com

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Abstract. On this work its develops the requirements to construct a compact k-dimensional manifold A without boundary since the scalar field defined with $a = \frac{l}{l'} = \frac{p\sqrt[n]{\pi^m}}{q\sqrt[n']{\pi^{m'}}}$ and establish the main scope about the trivial fundamental group on Poincare´s Conjecture for rational and irrational numbers.

Keyword: Poincare; Irrational; Manifold; Topological

[1]

1 Introduction

The original phrasing was as follows:

Consider a compact 3-dimensional manifold V without boundary. Is it possible that the fundamental group of V could be trivial, even though V is not homeomorphic to the 3-dimensional sphere?

To define the 3-dimensional manifold V; lets start by establish next properties about open sets and topological manifold.

Lets name:

$l = q\sqrt[n]{\pi^m} \in \Re$ ...(1)

and

$a = \frac{l}{l'} = \frac{p\sqrt[n]{\pi^m}}{q\sqrt[n']{\pi^{m'}}}$ ...(2)

[1]Independent Researcher- Non Affiliation; Mexico City; https://orcid.org/0009-0007-8175-7185, https://orcid.org/0009-0007-8175-7185

where $p, q, n, n', m, m' \in \Re$

Take into consideration that term given on (1) can be used to write some irrational numbers like the Gelfond- Schneider constant or Hilbert number when we match this expression such number; giving values to q, n and m.

i.e.

$2^{\sqrt{2}=q\ \sqrt[n]{\pi^m}}$

Clearing pi-root we could have:

$q = \frac{2^{\sqrt{2}}}{\sqrt[n]{\pi^m}}$

Hence q could be defined using different values of n and m and therefore Hilbert number could have different ways to express its value.

Now define **a** like the term of each coordinate on k-dimensions; therefore any point $a \in A$ on a k-dimensional space can be write as:

$$a \in \ R^k = (a_1, a_2, ..., a_k) = (\frac{p_1\ \sqrt[n_1]{\pi^{m_1}}}{q_1\ \sqrt[n'_1]{\pi^{m'_1}}}, \frac{p_2\ \sqrt[n_2]{\pi^{m_2}}}{q_2\ \sqrt[n'_2]{\pi^{m'_2}}}, ..., \frac{p_k\ \sqrt[n_k]{\pi^{m_k}}}{q_k\ \sqrt[n'_k]{\pi^{m'_k}}})$$

With this on mind lets construct a 3-dimensional manifold; starting by establish next three conditions to construct first a topological manifold.

1. Locally Euclidean
2. Hausdorff
3. Verify Second Countable Axiom

[1] A is Locally Euclidean

Lets prove that every point $a \in A$ has a neighbourhood which is homeomorphic to real k-space $\Re^k$ by establishing the next conditions for a function f between both topological spaces.

Remember that on Euclidean space $\Re^k$ the Euclidean topology is the natural topology induced by the Euclidean metric. Also take into consideration that to be homeomorphic both spaces need to be of same dimension.

[1.1] f is one-to-one i.e.

Take $f(a_1)$, $f(a_2) \in \Re^k$

Now take into consideration next properties about term given on (2)

·) For (2) we have with $m = m' = 0$ and $n, n' \in \Re$

$a = \frac{p}{q} \in \Re$

This should mean that any value that could have $f(a) \in R^k$ space can be described using early condition; including those cases when q is trivial.

··) To consider the possibility to establish more values associated with pi-roots on both topological spaces the term given for **a** will be used complete explicitly.

If $f(a_1) = f(a_2)$ then

Using term given on (2) on both sides of early condition we have:

$$f(a_1) = f((\frac{p_1 \sqrt[n_1]{\pi^{m_1}}}{q_1 \sqrt[n_1']{\pi^{m_1'}}})_{a_1}, ..., (\frac{p_k \sqrt[n_k]{\pi^{m_k}}}{q_k \sqrt[n_k']{\pi^{m_k'}}})_{a_1})$$

and

$$f(a_2) = f((\frac{p_1 \sqrt[n_1]{\pi^{m_1}}}{q_1 \sqrt[n_1']{\pi^{m_1'}}})_{a_2}, ..., (\frac{p_k \sqrt[n_k]{\pi^{m_k}}}{q_k \sqrt[n_k']{\pi^{m_k'}}})_{a_2})$$

by "if condition" then

$(\frac{p_1 \sqrt[n_1]{\pi^{m_1}}}{q_1 \sqrt[n_1']{\pi^{m_1'}}})_{a_1} = (\frac{p_1 \sqrt[n_1]{\pi^{m_1}}}{q_1 \sqrt[n_1']{\pi^{m_1'}}})_{a_2}$

.

.

$(\frac{p_k \sqrt[n_k]{\pi^{m_k}}}{q_k \sqrt[n_k']{\pi^{m_k'}}})_{a_2} = (\frac{p_k \sqrt[n_k]{\pi^{m_k}}}{q_k \sqrt[n_k']{\pi^{m_k'}}})_{a_2}$

therefore

$(a_1) = (a_2)$

Hence f is one-to-one

[1.2] f is onto

If

$$f(a) = f(\frac{p_1 \sqrt[n_1]{\pi^{m_1}}}{q_1 \sqrt[n'_1]{\pi^{m'_1}}} \ , \ ... \ , \ \frac{p_k \sqrt[n_k]{\pi^{m_k}}}{q_k \sqrt[n'_k]{\pi^{m'_k}}}) \in \Re^k$$

$$f(a) = (f_1(\frac{p_1 \sqrt[n_1]{\pi^{m_1}}}{q_1 \sqrt[n'_1]{\pi^{m'_1}}}) \ , \ ... \ , \ f_k(\frac{p_k \sqrt[n_k]{\pi^{m_k}}}{q_k \sqrt[n'_k]{\pi^{m'_k}}})) \in \Re^k$$

Applying inverse function f^{-1} we have:

$$a = (f_1^{-1}(f_1(\frac{p_1 \sqrt[n_1]{\pi^{m_1}}}{q_1 \sqrt[n'_1]{\pi^{m'_1}}})) \ , \ ... \ , \ f_k^{-1}(f_k(\frac{p_k \sqrt[n_k]{\pi^{m_k}}}{q_k \sqrt[n'_k]{\pi^{m'_k}}}))) \in A$$

Simplifying

$$a = (\frac{p_1 \sqrt[n_1]{\pi^{m_1}}}{q_1 \sqrt[n'_1]{\pi^{m'_1}}} \ , \ ... \ , \frac{p_k \sqrt[n_k]{\pi^{m_k}}}{q_k \sqrt[n'_k]{\pi^{m'_k}}}) \in A$$

Hence

$\exists \ a \in A \ \forall f(a) \in \Re^k$

Therefore

f is onto

Therefore f is a bijection

[1.3] f is continuous

Def.

A function

$f : X \to Y$

between two topological spaces X and Y is continuous if for every open set $V \subseteq Y$, the inverse image

$f^1(V) = \{x \in X | f(x) \in V\}$

is an open subset of X.

Proof.

Lets call $V \subseteq \Re^k$ the image of f

Like before

$f(a) = (f_1(\frac{p_1}{q_1}\frac{\sqrt[n_1]{\pi^{m_1}}}{\sqrt[n'_1]{\pi^{m'_1}}})_a \ , \ ... \ , \ f_k(\frac{p_k}{q_k}\frac{\sqrt[n_k]{\pi^{m_k}}}{\sqrt[n'_k]{\pi^{m'_k}}})_a) \in V$

Now consider $f(c), f(r) \in V$ such that

$f(c) = (f_1(\frac{p_1}{q_1}\frac{\sqrt[n_1]{\pi^{m_1}}}{\sqrt[n'_1]{\pi^{m'_1}}})_c \ , \ ... \ , \ f_k(\frac{p_k}{q_k}\frac{\sqrt[n_k]{\pi^{m_k}}}{\sqrt[n'_k]{\pi^{m'_k}}})_c) \in V$

and

$f(r) = (f_1(\frac{p_1}{q_1}\frac{\sqrt[n_1]{\pi^{m_1}}}{\sqrt[n'_1]{\pi^{m'_1}}})_r \ , \ ... \ , \ f_k(\frac{p_k}{q_k}\frac{\sqrt[n_k]{\pi^{m_k}}}{\sqrt[n'_k]{\pi^{m'_k}}})_r) \in V$

Using Euclidean distance to calculate length of interval between points **f(c)** and **f(r)** we have:

$\sqrt{(f_1(\frac{p_r}{q_r}\frac{\sqrt[n_r]{\pi^{m_r}}}{\sqrt[n'_r]{\pi^{m'_r}}})_r - f_1(\frac{p_c}{q_c}\frac{\sqrt[n_c]{\pi^{m_c}}}{\sqrt[n'_c]{\pi^{m'_c}}})_c)^2 + ... + (f_k(\frac{p_r}{q_r}\frac{\sqrt[n_r]{\pi^{m_r}}}{\sqrt[n'_r]{\pi^{m'_r}}})_r - f_k(\frac{p_c}{q_c}\frac{\sqrt[n_c]{\pi^{m_c}}}{\sqrt[n'_c]{\pi^{m'_c}}})_c)^2} = d(f(c), f(r)) \in V \subseteq \Re^k$

with $f(r) \neq f(c)$ then $d(f(c), f(r)) > 0$

Note that equation for $d(f(c), f(r))$ establish the square root of the equation of a k-sphere of radius equal ϵ

Lets take a point $f(a)$ into k-sphere and establish next fact by Law of Cosines:

$\sqrt{d(f(c), f(a))^2 + d(f(a), f(r))^2 - 2d(f(c), f(a)) \cdot d(f(a), f(r))cos < f(c)f(a)f(r) >} = d(f(c), f(r))$

where $< f(c)f(a)f(r) >$ is the angle formed by the three points indicated early.

Hence

$d(f(c), f(a)) < d(f(c), f(r))$

Like

$d(f(c), f(r)) = \epsilon$

Therefore

$f(a) \in V$ fulfilled condition $d(f(c), f(a)) < \epsilon$

Hence $\exists \ \epsilon$ that satisfied next condition to establish an open set in V

$B(f(c), f(r)) = \{f(a) \in V \text{ — } d(f(c), f(a)) < \epsilon\}$

sometimes named with etiquette $B_f(i)$

To fulfilled condition about V to be open now define next topological properties:

Let

$f(a) \in B(f(c_i), f(r_i))$ and $f(a) \in B(f(c_j), f(r_j))$

with $i \neq j$

then $f(a) \in B(f(c_i), f(r_i)) \cup B(f(c_j), f(r_j))$

Like $B_f(i)$ is open and $B_f(j)$ is open therefore union set is open

Besides if $f(a) = f(c_k)$ and $f(r_k) \in V$

$\exists\ \epsilon_k$ such that $d(f(c_k), f(a_k)) < \epsilon_k$

Therefore

$B(f(c_k), f(r_k)) = \{f(a_k) \in V \text{ — } d(f(c_k), f(a_k)) < \epsilon_k\}$

Let $f(a_k) \in B_f(k)$ and $f(a_k) \in B(f(c_l), f(r_l))$

Therefore

$f(a_k) \in B_f(k) \cup B_f(l)$

furthermore like $B_f(k)$ is open and $B_f(l)$ is open therefore union set is open.

Note this property maintains $\forall f(a) \in V$

Hence V is open by $\bigcup_{i \in I} B_{f(i)} \in V$

In the other hand note that same property also maintains for

$\bigcap_{i \in I} B_{f(i)} \in V$; $\forall f(a) \in V$

Hence V is open by $\bigcup_{i \in I} B_{f(i)} \in V$

Hence V is open.

Later if we take $f(a) \in V$ and apply inverse function, such that $f^{-1}(f(a)) \in f^{-1}(V)$ then we have for every $a \in A$ an associated open set in V by early procedure.

Note that as before by Cosine Law procedure made to establish an open set $B_(f(c), f(r)) \in V$; also maintains for $a \in B(c, r) \in W \subseteq A$. W is also named like the pre-image of f.

Even more; procedure to establish that V is open also maintains for W. Hence this means that W is open by $\bigcap_{i \in I} B_{f(i)} \in V$ and $\bigcup_{i \in I} B_i \in V$ therefore W is open.

Hence $\exists\ B_f(i) \in V; \forall a \in B_i \in W \subseteq A$

$\therefore$

$f^1(V) = \{a \in A \text{ — } f(a) \in V\}$

[1.4] f^{-1} is continuous

Procedure to prove that f is continuous is analogous for f^{-1}

Hence

$f^{-1}(W) = \{f(a) \in V \text{ — } a \in W\}$ is an open set of A.

[2] A is Hausdorff

Def.

Points x and y in a topological space X can be separated by neighbourhoods if there exists a neighbourhood U of x and a neighbourhood V of y such that U and V are disjoint $U \cap V = \varnothing$. X is a Hausdorff space if any two distinct points in X are separated by neighbourhoods.

Lets consider two open sets U and V on **A** such that disjoint property fulfill:

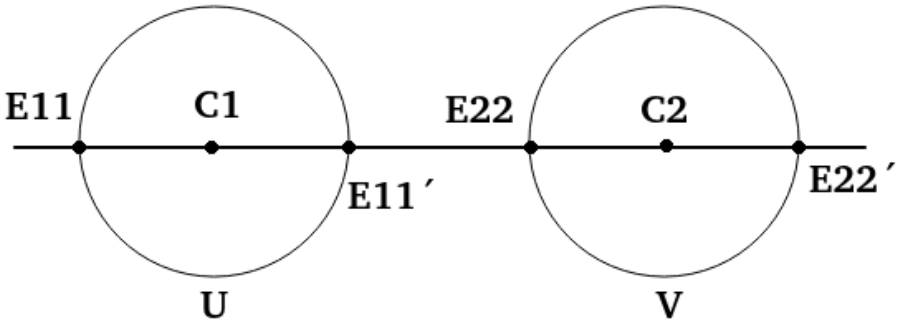

Diagram 1. Two open sets disjoint condition.

Lets consider the points showed on early diagram and define its distances:

$d(C_1, E_{11}) = r_1$

$d(C_1, E'_{11}) = r_1$

$d(C_2, E_{22}) = r_2$

$d(C_2, E'_{22}) = r_2$

Considering geometry requested by disjoint condition we have that:

$d(E_{11}, E'_{11}) = d_{max}$

$d(E_{22}, E'_{22}) = d_{min}$

Now taking two points $a_1 \in V \in A$ and $a_2 \in U \in A$

If as early:

$a_1 \in U = \{a_1 \in A \text{ — } d(a_1, c_1) < \epsilon_{11}\}$

and

$a_2 \in V = \{a_2 \in A | d(a_2, c_2) < \epsilon_{22}\}$

With ϵ_{xx} := open set radius

Hence

$d(a_1, a_2) < d_{max}$ and $d_{min} < d(a_1, a_2)$

Hence

$d_{min} < d(a_1, a_2) < d_{max}$

Therefore

A is Hausdorff

[3] Verify Second Countable Axiom

Def. a topological space T is second-countable if there exists some countable collection $U = \{U_i\}_{i=1}^{\infty}$ of open subsets of T such that any open subset of T can be written as a union of elements of some subfamily of U.

Proof. We have define early that the elements of the pre-image W are the open sets $a \in B(c, r) \in W \subseteq A$. To verify SCA is necessary to prove first that $B(c, r)$ can be a countable collection of opens in W.

Lets take $a, c \in \Re^k$ and define its distance expression as next:
$d(a, c) = \epsilon < r$

Being last also the condition to define an open set on A.

The collection of all distances defined on the pre-image W must satisfy trichotomy law considering another pair $a', c' \in A$

On the other hand by homeomorphism between A and $\Re^k$ we have specifically for onto property that:

if $f(c, a) = f(c', a')$ hence $(c, a) = (c', a')$

This imply that trichotomy only establish

$d(a, c) < d(a', c')$ or $d(a', c') < d(a, c)$

Because every distance that satisfy $d(a, c) = d(a', c')$ is established always between the same pairs of points taken on the pre-image W.

Under this condition the collection of all distances between pairs $a, c \in W$ can be indexed; i.e. $B(c, a)$ is a countable collection of open sets on A.

Lets name the i-esim countable open set on real line like:

$B_i(c, a) = \{a \in A \text{ — } d(c, a)_i = \epsilon_i < r_i\}$

If

$a \in B_i(c, a)$

hence

$d(c, a) < d(c, r)$

Left term establish next open set:

$B(c, a') = \{a' \in A \text{ — } d(c, a') < r' = d(c, a)\}$

renaming and considering trichotomy we will have:

$B(c, a') = \{a' \in A \text{ — } d(c, a')_{i-1} < r' = d(c, a)_i\}$

i.e.

$\forall a' \in B(c, a')$ is established

$d(c, a')_{i-1} < d(c, a)_i < d(c, r)$

and so on ...

Therefore

$B_1(c, a^*) \subset ... \subset B(c, a')_{i-1} \subset B(c, a)_i = \bigcup_{i=1}^{i} B_i = B_i$

i.e. any open subset of T can be written as a union of elements of some subfamily of U

[4 Differential Manifold]

[4.1 Cover for A]

To construct a differential manifold we have to define formally an atlas on the topological space A.

Def. An atlas for a topological space M is an indexed family $\{(U_\alpha, \varphi_\alpha) : \alpha \in I\}$ of charts on M which covers M (that is, $\bigcup_{\alpha \in I} U_\alpha = M$). If for some fixed n, the image of each chart is an open subset of n-dimensional Euclidean space, then M is said to be an n-dimensional manifold.

Proof.

A chart (B, f) on A consists of an open subset B of A, and a homeomorphism f from B to an open subset B_f of some Euclidean space R^n.

We have defined conditions to establish a chart with "**Locally Euclidean property**"; then is only necessary to define that the indexed family of open sets W (pre-image) cover A.

Lets take $W = Im(f) \subset A$ and $a \in A$ but $a \notin W$.

Establishing a second homeomorphism f' for $a \notin W$ we will have that: $\exists W'$ the pre-image of a second homeomorphism.

Let index both pre-images as W_i as well as both homeomorphism f_i with i = {1,2} and establish a third homeomorphism f_3 with $a \notin W_1$ and W_2.

Like before we will obtain another pre-image W_3. Lets do this procedure i times; such that:

$\exists a \notin W_i$ and $a \in A$ with such that an homeomorphism f_i will be defined.

Like this procedure exhaustively use every $a \in A$ with $a \notin W_i$ hence we can affirm that:

$\bigcup_{i=1}^{I} W_i = A$

Therefore the indexed family W_i is a cover for A

Besides (W_i, f_i) is an atlas for A

[4.2 Transition Map is Differentiable]

Def. Given any $\alpha, \beta \in \Lambda$ the application $\tau_{\alpha,\beta} = \varphi_\beta \circ \varphi_\alpha^{-1} : \varphi_\alpha(U_\alpha \cap U_\beta) \longrightarrow \varphi_\beta(U_\alpha \cap U_\beta)$, called "**Transition Map**" is differentiable of order r.

Def. A real valued function f on an n-dimensional differentiable manifold M is called differentiable at a point $p \in M$ if (U, ϕ) is a differentiable chart where U is an open set in M containing p and $\phi : U \to \mathbf{R}^n$ is the map defining the chart, then f is differentiable at p if and only if

$f \circ \phi^{-1} \colon \phi(U) \subset \mathbf{R}^n \to \mathbf{R}$

is differentiable at $\phi(p)$, that is $f \circ \phi^{-1}$ is a differentiable function from the open set $\phi(U)$, considered as a subset of $\mathbf{R}^n$ to $\mathbf{R}$.

Def. A function of several real variables $f : R^m \to R^n$ is said to be differentiable at a point x_0 if there exists a linear map $J : R^m \to R^n$ such that:

$lim_{h\to 0}\frac{||f(x_0+h)-f(x_0)-J(h)||_{R^n}}{||h||_{R^n}} = 0$

with

$$J = \begin{bmatrix} \dfrac{\partial \mathbf{f}}{\partial x_1} & \cdots & \dfrac{\partial \mathbf{f}}{\partial x_n} \end{bmatrix} = \begin{bmatrix} \nabla^{\mathrm{T}} f_1 \\ \vdots \\ \nabla^{\mathrm{T}} f_m \end{bmatrix} = \begin{bmatrix} \dfrac{\partial f_1}{\partial x_1} & \cdots & \dfrac{\partial f_1}{\partial x_n} \\ \vdots & \ddots & \vdots \\ \dfrac{\partial f_m}{\partial x_1} & \cdots & \dfrac{\partial f_m}{\partial x_n} \end{bmatrix}$$

Proof. Let be (U_α, ϕ_α) ; (U_β, ϕ_β) ; (U_ϕ, ϕ) with ϕ_α, ϕ_β, ϕ differentiable functions and consider next definition:

Def. Let $A \subseteq \Re^n$, $a \in A$ and $f : A \to \Re^n$ injective, differentiable in a and such that $f(a) \in f(A)$.

Then f^{-1} is differentiable in $f(a)$ if and only if f^{-1} is continuous on $f(a)$ and $Df(a)$ is reversible that is, $det|J(f(a))| \neq 0$

In such case $Df^{-1}(a) = |Df(a)|^{-1}$.

applying last definition to ϕ_α^{-1} we can affirm that $(U_\alpha, \phi_\alpha^{-1})$ is a differentiable chart.

Now consider the composition function:

$\phi_\beta \circ \phi_\alpha^{-1} \circ \phi = \tau_{\alpha,\beta} \circ \phi$

Such that $\phi(U_\alpha \bigcap U_\beta) \subset \mathbf{R}^n \to \mathbf{R}$

Like ϕ_α is an homeomorphism by former facts about f homeomorphism; ϕ_α^{-1} is an homeomorphism.

Like ϕ_α^{-1}, ϕ_β, ϕ are homeomorphism hence $\phi_\beta \circ \phi_\alpha^{-1} \circ \phi = \tau_{\alpha,\beta} \circ \phi$ is an homeomorphism.

A continuous function is not necessary differentiable; but a differentiable function is necessarily continuous.

Besides remember that the composition of differentiable functions is differentiable.

By homeomorphism property about composition function; we can not conclude that $\tau_{\alpha,\beta} \circ \phi$ is differentiable.

However like we are requesting that $\phi_\alpha, \phi_\beta, \phi$ being differentiable hence composition $\phi_\beta \circ \phi_\alpha^{-1} \circ \phi = \tau_{\alpha,\beta} \circ \phi$ is differentiable.

Therefore $\phi_\beta \circ \phi_\alpha^{-1} = \tau_{\alpha,\beta}$ is differentiable of order r = n-1.

This means that all the partial derivatives of a function exist in a neighborhood of a point x_0 and are continuous at the point x_0, then the function is differentiable at that point x_0.

Hence the linear map J is given by the Jacobian matrix, an $n \times m$ matrix in this case.

[4.3] M manifolds is compact

Def. A compact manifold is a manifold that is compact as a topological space.

Def. This form of compactness holds for closed and bounded subsets of Euclidean space and is known as the Heine–Borel theorem.

Def. "Heine Borel Theorem". For a subset S of Euclidean space $\mathbf{R}^n$, the following two statements are equivalent:

S is closed and bounded S is compact, that is, every open cover of S has a finite subcover.

Proof.

To fulfilled the condition to be compact is enough to prove that A is closed and bounded.

Def. A subset W of a topological space (A, τ) is called closed if its complement AnW is an open subset of (A, τ); that is, if $AnW \in \tau$.

We have defined before that $\bigcup_{i=1}^{I} W_i$ cover A; where $W_i' \in A$ is an open set and $1 \leq i' \leq I$.

Lets define

$A \backslash W_i' = \bigcup_{i=1}^{I} W_i \backslash W_i'$

taking some $i' \in I$ we have:

$\bigcup_{i=1}^{i'-1} W_i \cup \bigcup_{i'+1}^{I} W_i = A'$

Like W_i is open $\forall i \in [1, i'-1]$ hence $\bigcup_{i=1}^{i'-1} W_i$ is open.

Like W_i is open $\forall i \in [i'+1, I]$ hence $\bigcup_{i'+1}^{I} W_i$ is open.

Hence A' is open.

Therefore A' complement is closed.

Def. A set A in a metric space (A,d) is bounded if it has a finite generalized diameter, i.e., there is an $R < \infty$ such that $d(x, y) \leq R$ for all $x, y \in A$. A set in R^n is bounded iff it is contained inside some ball $x_1^2 + ... + x_n^2 \leq R^2$ of finite radius R.

Proof. Like we have been defined before an open set $W_i \in A$ like an open ball such that its radius R is finite; i.e. R exists; hence A is bounded.

Therefore closed and bounded condition on Heine-Borel theorem is satisfied for A.

Hence A is compact.

Hence A manifold is compact.

[5.1] M manifold has no boundary.

Def 1. Let M be an n-dimensional manifold with boundary. Among the charts (U, ϕ) of (the maximal atlas of) M there are charts with U open in R^n (that is, disjoint from $R^{n-1} \subset R^n$). The union of images of such charts is denoted as Int M; it is a dense open subset of M, and it is an n-dimensional manifold in the sense of our previous definition. It is called the interior of M . The difference M - Int M = δM is an (n-1)-dimensional manifold if $\{(U, \phi)\}$ is an atlas of M, then $\{(U \cap R^{n-1}, \phi\,|_{U \cap R^{n-1}})\}$ is an atlas of δM . The manifold δM is called the boundary of M.

Def 2. A point p ∈ M belongs to δM (the boundary of M) if and only if there is a chart (U, ϕ) of M such that $p = \phi(q)$ for some $q \in U \cap R^{n-1}$. (In particular, $\delta R^n = R^{n-1}$.)

Def 3. M manifold without boundary fulfilled

M - Int M = δM = { } := empty set

[**5.2 Poincares´ Conjecture**] If M a compact 3-dimensional manifold V without boundary is not homeomorphic to the 3-dimensional sphere then the fundamental group of V is trivial.

Proof. To be not homeomorphic establish next condition between the compact 3-dimensional manifold A without boundary and the 3-dimensional sphere:

A is non Locally Euclidean to the 3-dimensional sphere; i.e. every point $a' \in A$ has a neighbourhood which is not homeomorphic to the real 3-dimensional sphere.

Remember that in topology, the n-sphere is an example of a compact topological manifold without boundary.

This means that $\nexists$ g function such that g is bijective, continuous and inverse continuous between the open sets of both topological spaces.

Hence taking $g(a'), g(b') \in$ A a compact 3-dimensional manifold without boundary; we will have that if $g(a') = g(b')$ therefore $a' \neq b'$, $\forall$ a, b $\in$ 3-dimensional sphere.

Now lets take next definitions:

Def 1. The set of homotopy classes of loops in X based at "a" forms a group under the multiplication $< \alpha > \cdot < \beta > = < \alpha \cdot \beta >$

Def 2. Let $f, g : X \to Y$ be maps. Then f is homotopic to g if there exists a map $F : X \times I \to Y$ such that $F(x, 0) = f(x)$ and $F(x, 1) = g(x)$ for all points $x \in X$.

Def 3. By a loop in a space X we shall understand a map $\alpha : I \to X$ such that $\alpha(0) = \alpha(1)$, † and we shall say that the loop is based at the point $\alpha(0)$. If α and β are two loops based at the same point of X, we define the product $\alpha \cdot \beta$ to be the loop given by the formula:

$$\alpha \cdot \beta = \begin{cases} \alpha(2s); 0 \leq s \leq \frac{1}{2} \\ \beta(2s-1); \frac{1}{s} \leq s \leq 1 \end{cases}$$

† A slight change from the terminology used when a loop was a map from a circle to X.

Def 4. The identity element is the homotopy class of the constant loop e at a defined by $e(s) = a$ for $0 \leq s \leq 1$.

By the requirement to be not homeomorphic between both topological spaces: A (compact 3-dimensional manifold without boundary) and 3-dimensional sphere; we have that f, g established on def 2 not exist.

Hence by injective property about such homeomorphism if $g(a') = g(b')$ (on the cover form by the open sets defined on early procedure to establish the image of ϕ map); $\forall g(a'), g(b') \in$ A; a compact 3-dimensional manifold without boundary; only can be true if $g(s) = e(s)$ with $s \in A$.

Therefore the fundamental group of A a 3-dimensional manifold without boundary is trivial .

Remember that the loop e at "a" such that $e(s) = a$ is established $\forall a \in \Re$ with

$$a = \frac{l}{l'} = \frac{p\sqrt[n]{\pi^m}}{q\sqrt[n']{\pi^{m'}}}$$

i.e. is valid for every rational and irrational number on the field of $\Re$.

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2 References

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3 Additional References

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